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New braid group representations of the B_2, B_3 and B_4 types, their associated link polynomials and quantum R matrices

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Abstract. New braid group representations of the B_2, B_3 and B_4 types are obtained by solving the defining relations of Artin's braid group $\mathbb{B}_n$ directly; we give their associated link polynomials. We discuss a procedure (Baxterization) which allows us to construct their corresponding quantum R matrices.

1. Introduction

The quantum Yang-Baxter equation (QYBE)

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x) \tag{1.1}$$

introduced in [1, 2] plays a central role in the theory of completely integrable classical and quantum systems and in the theory of exactly solvable models in statistical mechanics [3-7]. If V is a complex vector space and $R(x) \in \mathrm{End}(V\otimes V)$ then $R_{ij}(x) \in \mathrm{End}(V\otimes V\otimes V)$ is a matrix that acts as $R(x)$ on the ith and jth spaces and as the identity on the remaining space; $R(x)$ is referred to as the quantum R matrix and $x \in \mathbb{C}$ is the multiplicative spectral parameter. Solutions of the QYBE have been obtained by solving (1.1) directly [5, 8-10] or by using more systematic algebraic approaches [3, 11-14].

In this paper we construct new solutions of the QYBE by exploiting an interesting connection between solutions of (1.1) and representations of Artin's braid group $\mathbb{B}_n$. This relationship can be seen by considering the following form of the QYBE:

$$(\check{R}(x)\otimes I)(I\otimes\check{R}(xy))(\check{R}(y)\otimes I) = (I\otimes\check{R}(y))(\check{R}(xy)\otimes I)(I\otimes\check{R}(x)) \tag{1.2a}$$

with

$$\check{R}(x) = PR(x) \tag{1.2b}$$

where $P \in \mathrm{End}(V\otimes V)$ denotes the transposition $u\otimes v \to v\otimes u$ and $I \in \mathrm{End}(V)$ is the identity matrix. Note that without the spectral parameter (1.2a) is one of the defining relations of $\mathbb{B}_n$; the matrix $S \equiv \check{R}(0)$ constitutes a representation of $\mathbb{B}_2$ and can be used to construct representations of $\mathbb{B}_n$ for any n. Given a solution $\check{R}(x)$, one can therefore extract from it braid group representations; this has lead to important developments in knot theory [5]. Some time ago [15] we observed that under certain restrictions it proves relatively easy to solve the defining relations of $\mathbb{B}_n$ directly. This has lead to a new infinite family of representations of $\mathbb{B}_n$ [16]. Exploiting this fact, our strategy for finding new solutions of the QYBE is the following. First we find new braid group representations and then proceed to transform them into quantum R matrices. This transformation is known as Baxterization [17].

Our paper is organized as follows. In section 2 we obtain new braid group representations of the B_2, B_3 and B_4 types and construct their associated link polynomials. In section 3, we proceed to Baxterize the new solutions of the B_2 and B_3 types. We conclude with a few remarks.

2. New braid group representations and their associated link polynomials

2.1. Artin's braid group and standard representations

$\mathbb{B}_n$ [18, 19] is generated by a set of $(n-1)$ generators $g_1, g_2, \ldots, g_{n-1}$ and their inverses subject to the following necessary and sufficient defining relations:

$$g_i g_j = g_j g_i \qquad |i-j| \geqslant 2 \tag{2.1a}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}. \tag{2.1b}$$

Let V be an N-dimensional vector space and $S \in \mathrm{End}(V \otimes V)$ be an $N^2 \times N^2$ matrix that has an inverse. The following mapping is a representation of $\mathbb{B}_n$:

$$\rho : \mathbb{B}_n \to \mathrm{End}(V^{\otimes n}) \qquad \rho(g_i) = I_1 \otimes \ldots \otimes I_{i-1} \otimes S \otimes I_{i+2} \ldots \otimes I_n \tag{2.2}$$

where the subscript i means that the ith vector space in $V^{\otimes n}$ and S acts in the ith and $(i+1)$th vector spaces. The form of (2.2) insures the satisfaction of (2.1a); no restriction needs to be imposed on S. The satisfaction of (2.1b) requires that S be a solution of

$$(S \otimes I)(I \otimes S)(S \otimes I) = (I \otimes S)(S \otimes I)(I \otimes S). \tag{2.3}$$

Our objective is to find new solutions of (2.3); our reference point will be the solutions which can be extracted from Bazhanov and Jimbo's quasi-classical quantum R matrices [13, 14] and which we will refer to as the standard solutions. Reshetikhin [20] has shown that the underlying mathematical structures behind these standard solutions are the quantized universal enveloping algebras U_q of simple Lie algebras; they can be generated by restricting the universal $\mathcal{R}$-matrix, which lies in $U_q \otimes U_q$, to fundamental representations Λ of U_q. For generic values of the deformation parameter q, the representation theory of U_q is the same as for the classical $(q=1)$ case [21]. It follows that these standard solutions decompose according to the classical decomposition rule of direct products of irreducible representations

$$\Lambda \otimes \Lambda = \sum_{i=1}^{l} \phi_i. \tag{2.4a}$$

Their spectra decomposition, characteristic polynomial $\Delta(\lambda)$ and minimal polynomial $m(\lambda)$ are consistent with the decomposition rule (2.4a)

$$\begin{aligned} S &= \sum_{i=1}^{l} \lambda_i \mathbb{P}_i \qquad \Delta(\lambda) = (\lambda - \lambda_1)^{f_1} \ldots (\lambda - \lambda_l)^{f_l} \\ m(\lambda) &= (\lambda - \lambda_1) \ldots (\lambda - \lambda_l) \end{aligned} \tag{2.4b}$$

where the λ_i are the l distinct eigenvalues of S, f_i is the dimension of the irreducible representation ϕ_i in (2.4a) and the $\mathbb{P}_i$ are the projectors. We now turn to the problem of finding new solutions of (2.3) whose main characteristic is that their decomposition rule does not follow the rule (2.4a) of the classical case; they will be referred to as non-standard solutions.

2.2. *Non-standard solutions of the B_2 type*

Our starting point is the solution of (2.3) which can be extracted from Bazhanov and Jimbo's $B_2^{(1)}$ quantum R matrix [13, 14]. This solution is associated with the fundamental irreducible representation of B_2, and its block structure is as follows:

$$S = \text{block diag}(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_{-4}, \tau_{-3}, \tau_{-2}, \tau_{-1})$$

$$\tau_1 = z_1 \qquad \tau_2 = \begin{pmatrix} 0 & z_2 \\ z_2 & z_3 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 0 & 0 & z_4 \\ 0 & z_5 & 0 \\ z_4 & 0 & z_6 \end{pmatrix}$$

$$\tau_4 = \begin{pmatrix} 0 & 0 & 0 & z_7 \\ 0 & 0 & z_8 & 0 \\ 0 & z_8 & z_9 & 0 \\ z_7 & 0 & 0 & z_{10} \end{pmatrix} \qquad \tau_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & z_{11} \\ 0 & 0 & 0 & z_{12} & z_{13} \\ 0 & 0 & z_{14} & z_{15} & z_{16} \\ 0 & z_{12} & z_{15} & z_{17} & z_{18} \\ z_{11} & z_{13} & z_{16} & z_{18} & z_{19} \end{pmatrix} \qquad z_i \neq 0 \text{ all } i \tag{2.5}$$

$$\tau_{-4} = \begin{pmatrix} 0 & 0 & 0 & z_{20} \\ 0 & 0 & z_{21} & 0 \\ 0 & z_{21} & z_{22} & 0 \\ z_{20} & 0 & 0 & z_{23} \end{pmatrix} \qquad \tau_{-3} = \begin{pmatrix} 0 & 0 & z_{24} \\ 0 & z_{25} & 0 \\ z_{24} & 0 & z_{26} \end{pmatrix}$$

$$\tau_{-2} = \begin{pmatrix} 0 & z_{27} \\ z_{27} & z_{28} \end{pmatrix} \qquad \tau_{-1} = z_{29}.$$

Solutions with the block structure described in (2.5) will be referred to as solutions of the B_2 type. The question we addressed is the following: is the standard solution the only solution of the B_2 type? We approached this problem by solving (2.3) directly. The method used is an extension of the one described in [15] and has already led to non-standard solutions of the A_1, C_2, D_2 and D_3 types [16, 22, 23]; it consists in obtaining solutions of (2.3) by solving a minimal subset of these equations and then verifying, using a symbolic manipulation computer code [24], that all the equations are satisfied. There exist only two distinct solutions of the B_2 type, which we denote S and $\tilde{S}$. S is the standard solution and is as follows:

$$S = \text{block diag}(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_{-4}, \tau_{-3}, \tau_{-2}, \tau_{-1})$$

$$\tau_1 = \tau_{-1} = q \qquad \tau_2 = \tau_{-2} = \begin{pmatrix} 0 & 1 \\ & w \end{pmatrix} \qquad \tau_3 = \tau_{-3} = \begin{pmatrix} 0 & 0 & 1 \\ & q & 0 \\ & & w \end{pmatrix}$$

$$\tau_4 = \tau_{-4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 1 & 0 \\ & & w & 0 \\ & & 0 & w \end{pmatrix} \qquad \tau_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & q^{-1} \\ & 0 & 0 & q^{-1} & -q^{-1}w \\ & & 1 & -q^{-1/2}w & -q^{-3/2}w \\ & & & (1-q^{-1})w & -q^{-2}w \\ & & & & (1-q^{-3})w \end{pmatrix} \tag{2.6a}$$

where $w \equiv q - q^{-1}$ and all submatrices $\tau_{\pm i}$ are symmetric. The spectral decomposition, characteristic and minimal polynomials of S are

$$\lambda_1 = q \qquad \lambda_2 = -q^{-1} \qquad \lambda_3 = q^{-4}$$
$$S = \sum_{i=1}^{3} \lambda_i \mathbb{P}_i \tag{2.6b}$$
$$\Delta(\lambda) = (\lambda-\lambda_1)^{14}(\lambda-\lambda_2)^{10}(\lambda-\lambda_3)$$
$$m(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).$$

S is equal to the solution extracted from Jimbo's $B_2^{(1)}$ quantum R matrix (equation (3.6) in [14]) within an overall factor q^4. Note that the decomposition follows the classical case $(5)\times(5) = (14)+(10)+(1)$. The second solution is as follows [22]:

$$\tilde{S} = \text{block diag}(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4, \tilde{\tau}_5, \tilde{\tau}_{-4}, \tilde{\tau}_{-3}, \tilde{\tau}_{-2}, \tilde{\tau}_{-1})$$
$$\tilde{\tau}_1 = \tilde{\tau}_{-1} = \tau_1 \qquad \tilde{\tau}_2 = \tilde{\tau}_{-2} = \tau_2 \qquad \tilde{\tau}_3 = \tilde{\tau}_{-3} = \begin{pmatrix} 0 & 0 & 1 \\ & -q^{-1} & 0 \\ & & w \end{pmatrix}$$
$$\tilde{\tau}_4 = \tilde{\tau}_{-4} = \tau_4 \qquad \tilde{\tau}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & q^{-1} \\ & 0 & 0 & -q & \mathrm{i}w \\ & & 1 & +\mathrm{i}q^{1/2}w & q^{1/2}w \\ & & & (1+q)w & -\mathrm{i}qw \\ & & & & (1-q)w \end{pmatrix} \tag{2.7a}$$

with $w \equiv q - q^{-1}$ and all $\tilde{\tau}_{\pm i}$ are symmetric. $\tilde{S}$ is a non-standard solution which distinguishes itself from S by its different distinct eigenvalues and decomposition rule

$$\lambda_1 = q \qquad \lambda_2 = -q^{-1} \qquad \lambda_3 = 1$$
$$\tilde{S} = \sum_{i=1}^{3} \lambda_i \tilde{\mathbb{P}}_i \tag{2.7b}$$
$$\Delta(\lambda) = (\lambda-\lambda_1)^{12}(\lambda-\lambda_2)^{12}(\lambda-\lambda_3)$$
$$m(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).$$

2.3. *Non-standard solution of the B_3 type*

For solutions of the B_3 type, the reference point is the standard solution extracted from the $B_3^{(1)}$ quantum R matrix [14] and which is associated with the fundamental irreducible presentation of B_3; using the same strategy as for B_2, we found that there exist only four solutions of the B_3 type. In addition to the standard solution there exist three non-standard solutions which are related by similarity transformations (they have the same characteristic and minimal polynomials). The standard solution, which we denote S, is as follows:

$$S = \text{block diag}\,(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_{-6}, \gamma_{-5}, \gamma_{-4}, \gamma_{-3}, \gamma_{-2}, \gamma_{-1})$$

where all submatrices are symmetric:

$$\gamma_1 = \gamma_{-1} = q \qquad \gamma_2 = \gamma_{-2} = \tau_2 \qquad \gamma_3 = \gamma_{-3} = \tau_3 \qquad \gamma = \gamma_{-4} = \tau_4$$

$$\gamma_5(\alpha)=\gamma_{-5}(\alpha)=\begin{pmatrix}0&0&0&0&1\\ &0&0&1&0\\ &&\alpha&0&0\\ &&&w&0\\ &&&&w\end{pmatrix}\qquad \gamma_6=\gamma_{-6}=\begin{pmatrix}0&0&0&0&0&1\\ &0&0&0&1&0\\ &&0&1&0&0\\ &&&w&0&0\\ &&&&w&0\\ &&&&&w\end{pmatrix}$$

$$\gamma_7=\begin{pmatrix}0&0&0&0&0&0&q^{-1}\\ &0&0&0&0&q^{-1}&q^{-1}w\\ &&0&0&q^{-1}&q^{-1}w&-q^{-2}w\\ &&&1&q^{-1/2}w&-q^{-3/2}w&q^{-5/2}w\\ &&&&-q^{-1}(1-q)w&q^{-2}w&-q^{-3}w\\ &&&&&(1-q^{-3})w&q^{-4}w\\ &&&&&&(1-q^{-5})w\end{pmatrix} \tag{2.8a}$$

where $w\equiv q-q^{-1}$, $\alpha=q$ and the τ_i are given in (2.6a); all $\gamma_{\pm i}$ are symmetric. Note that in γ_0 our choice of signs differs from that of Jimbo. In addition we have that

$$\begin{aligned}&\lambda_1=q\qquad \lambda_2=-q^{-1}\qquad \lambda_3=q^{-6}\\ &S=\sum_{i=1}^{3}\lambda_i\mathbb{P}_i\\ &\Delta(\lambda)=(\lambda-\lambda_1)^{27}(\lambda-\lambda_2)^{21}(\lambda-\lambda_3)\\ &m(k)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\end{aligned} \tag{2.8b}$$

note that S follows the classical decomposition rule $(7)\times(7)=(27)+(21)+(1)$. We shall give only one of the non-standard solutions:

$$\tilde S=\text{block diag}\,(\tilde\gamma_1,\tilde\gamma_2,\tilde\gamma_3,\tilde\gamma_4,\tilde\gamma_5,\tilde\gamma_6,\tilde\gamma_7,\tilde\gamma_{-6},\tilde\gamma_{-5},\tilde\gamma_{-4},\tilde\gamma_{-3},\tilde\gamma_{-2},\tilde\gamma_{-1})$$

$$\tilde\gamma_1=\tilde\gamma_{-1}=q\qquad \tilde\gamma_2=\tilde\gamma_{-2}=\tau_2\qquad \tilde\gamma_3=\tilde\gamma_{-3}=\tau_3\qquad \tilde\gamma_4=\tilde\gamma_{-4}=\tau_4$$

$$\tilde\gamma_5=\tilde\gamma_{-5}=\gamma_5(\alpha=-q^{-1})\qquad \tilde\gamma_6=\tilde\gamma_{-6}=\gamma_6$$

$$\tilde\gamma_7=\begin{pmatrix}0&0&0&0&0&0&q^{-1}\\ &0&0&0&0&q^{-1}&q^{-1}w\\ &&0&0&-q&\mathrm{i}w&-\mathrm{i}q^{-1}w\\ &&&1&\mathrm{i}q^{1/2}w&q^{1/2}w&-q^{-1/2}w\\ &&&&(1+q)w&-\mathrm{i}qw&\mathrm{i}w\\ &&&&&(1-q)w&w\\ &&&&&&(1-q^{-1})w\end{pmatrix} \tag{2.9a}$$

where $w\equiv q-q^{-1}$ and where all submatrices $\tilde\gamma_{\pm i}$ are symmetric. In addition we have that

$$\begin{aligned}&\lambda_1=q\qquad \lambda_2=-q^{-1}\qquad \lambda_3'=q^{-2}\\ &\tilde S=\sum_{i=1}^{3}\lambda_i\tilde{\mathbb{P}}_i\\ &\Delta(\lambda)=(\lambda-\lambda_1)^{25}(\lambda-\lambda_2)^{23}(\lambda-\lambda_3)\\ &m(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).\end{aligned} \tag{2.9b}$$

Note that $\tilde S$ does not follow the classical decomposition rule.

2.4. Non-standard solutions of the B_4 type

Our reference point is the solution of (2.3), which can be extracted from the $B_4^{(1)}$ quantum R matrix [14] and which is associated with the fundamental irreducible representation of B_4; we refer to this solution as the standard one of the B_4 type. Our strategy is the same as in the previous cases and we have looked for new solutions with the same block structure as the standard one. We have found that there are only eight solutions; in addition to the standard solutions, there exist seven non-standard solutions; an eigenvalue analysis reveals that these seven solutions divide into two equivalence classes (solutions within a class are related by a similarity transformation). We will give a representative of each class. The standard solution is as follows:

$$S = \text{block diag}\,(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{-8}, \pi_{-7}, \pi_{-6}, \pi_{-5}, \pi_{-4}, \pi_{-3}, \pi_{-2}, \pi_{-1})$$

$$\pi_1 = \pi_{-1} = q \qquad \pi_2 = \pi_{-2} = \tau_2 \qquad \pi_3 = \pi_{-3} = \tau_3 \qquad \pi_4 = \pi_{-4} = \tau_4$$

$$\pi_5 = \pi_{-5} = \gamma_5(\alpha = q) \qquad \pi_6 = \pi_{-6} = \gamma_6$$

$$\pi_7(\beta) = \pi_{-7}(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 & 0 & 0 \\ & & & \beta & 0 & 0 & 0 \\ & & & & w & 0 & 0 \\ & & & & & w & 0 \\ & & & & & & w \end{pmatrix}$$

$$\pi_8 = \pi_{-8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 & 0 & 0 \\ & & & & w & 0 & 0 & 0 \\ & & & & & w & 0 & 0 \\ & & & & & & w & 0 \\ & & & & & & & w \end{pmatrix} \tag{2.10a}$$

$$\pi_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \\ & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & -q^{-1}w \\ & & 0 & 0 & 0 & 0 & q^{-1} & -q^{-1}w & -q^{-2}w \\ & & & 0 & 0 & q^{-1} & -q^{-1}w & -q^{-2}w & -q^{-3}w \\ & & & & 1 & -q^{-1/2}w & -q^{-3/2}w & -q^{-5/2}w & -q^{-7/2}w \\ & & & & & (1-q^{-1})w & -q^{-2}w & -q^{-3}w & -q^{-4}w \\ & & & & & & (1-q^{-3})w & -q^{-4}w & -q^{-5}w \\ & & & & & & & (1-q^{-5})w & -q^{-6}w \\ & & & & & & & & (1-q^{-7})w \end{pmatrix}$$

where $w \equiv q - q^{-1}$, $\beta = q$ and γ_i and τ_i are given in (2.8a) and (2.6a), respectively. S has the following properties:

$$\lambda_1 = q \qquad \lambda_2 = -q^{-1} \qquad \lambda_3 = q^{-8}$$

$$S = \sum_{i=1}^{3} \lambda_i \mathbb{P}_i \tag{2.10b}$$

$$\Delta(\lambda) = (\lambda - q)^{44} (\lambda + q^{-1})^{36} (\lambda - q^{-8})$$

which is in accordance with the classical decomposition rule $(9) \times (9) = (44) + (36) + (1)$.

The second solution $\tilde{S}$ is of the non-standard type:

$$\tilde{S} = \text{block diag}\,(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4, \tilde{\pi}_5, \tilde{\pi}_6, \tilde{\pi}_7, \tilde{\pi}_8, \tilde{\pi}_9, \tilde{\pi}_{-8}, \tilde{\pi}_{-7}, \tilde{\pi}_{-6}, \tilde{\pi}_{-5}, \tilde{\pi}_{-4}, \tilde{\pi}_{-3}, \tilde{\pi}_{-2}, \tilde{\pi}_{-1})$$

$$\tilde{\pi}_1 = \tilde{\pi}_{-1} = q \qquad \tilde{\pi}_2 = \tilde{\pi}_{-2} = \tau_2 \qquad \tilde{\pi}_3 = \tilde{\pi}_{-3} = \tau_3 \qquad \tilde{\pi}_4 = \tilde{\pi}_{-4} = \tau_4$$

$$\tilde{\pi}_5 = \tilde{\pi}_{-5} = \gamma_5(\alpha = q) \qquad \tilde{\pi}_6 = \tilde{\pi}_{-6} = \gamma_6$$

$$\tilde{\pi}_7 = \tilde{\pi}_{-7} = \pi_7(\beta = -q^{-1}) \qquad \tilde{\pi}_8 = \tilde{\pi}_{-8} = \pi_8 \tag{2.11a}$$

$$\tilde{\pi}_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \\ & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & q^{-1}w \\ & & 0 & 0 & 0 & 0 & q^{-1} & q^{-1}w & -q^{-2}w \\ & & & 0 & 0 & -q & \mathrm{i}w & -\mathrm{i}q^{-1}w & \mathrm{i}q^{-2}w \\ & & & & 1 & \mathrm{i}q^{1/2}w & q^{1/2}w & -q^{-1/2}w & q^{-3/2}w \\ & & & & & (1+q)w & -\mathrm{i}qw & \mathrm{i}w & -\mathrm{i}q^{-1}w \\ & & & & & & (1-q)w & w & -q^{-1}w \\ & & & & & & & (1-q^{-1})w & q^{-2}w \\ & & & & & & & & (1-q^{-3})w \end{pmatrix}$$

where $w \equiv q - q^{-1}$ and the τ_i and γ_i are given in (2.6a) and (2.8a), respectively. $\tilde{S}$ has the following properties:

$$\lambda_1 = q \qquad \lambda_2 = -q^{-1} \qquad \lambda_3 = q^{-4}$$

$$\tilde{S} = \sum_{i=1}^{3} \lambda_i \tilde{\mathbb{P}}_i$$

$$\Delta(\lambda) = (\lambda - \lambda_1)^{42} (\lambda - \lambda_2)^{38} (\lambda - \lambda_3) \tag{2.11b}$$

$$m(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

The third solution is also of the non-standard type

$$S^* = \text{block diag}\,(\pi_1^*, \pi_2^*, \pi_3^*, \pi_4^*, \pi_5^*, \pi_6^*, \pi_7^*, \pi_8^*, \pi_9^*, \pi_{-8}^*, \pi_{-7}^*, \pi_{-6}^*, \pi_{-5}^*, \pi_{-4}^*, \pi_{-3}^*, \pi_{-2}^*, \pi_{-1}^*)$$

$$\pi_1^* = \pi_{-1}^* = q \qquad \pi_2^* = \pi_{-2}^* = \tau_2 \qquad \pi_3^* = \pi_{-3}^* = \tau_3 \qquad \pi_4^* = \pi_{-4}^* = \tau_4$$

$$\pi_5^* = \pi_{-5}^* = \gamma_5(\alpha = -q^{-1}) \qquad \pi_6^* = \pi_{-6}^* = \gamma_6$$

$$\pi_7^* = \pi_{-7}^* = \pi_7(\beta = -q^{-1}) \qquad \pi_8^* = \pi_{-8}^* = \pi_8$$

$$\pi_9^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \\ & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & q^{-1}w \\ & & 0 & 0 & 0 & 0 & -q & iw & -iq^{-1}w \\ & & & 0 & 0 & -q & -qw & iqw & -iw \\ & & & & 1 & iq^{1/2}w & iq^{3/2}w & q^{3/2}w & -q^{1/2}w \\ & & & & & (1+q)w & q^2w & -iq^2w & iqw \\ & & & & & & (1+q^3)w & -iq^3w & iq^2w \\ & & & & & & & (1-q^3)w & q^2w \\ & & & & & & & & (1-q)w \end{pmatrix} \tag{2.12a}$$

with $w \equiv q - q^{-1}$ and the τ_i and γ_i are given in (2.6a) and (2.8a), respectively. S^* has the following properties:

$$\begin{aligned} &\lambda_1 = q \qquad \lambda_2 = -q^{-1} \qquad \lambda_3 = 1 \\ &S^* = \sum_{i=1}^{3} \lambda_i \mathbb{P}_i^* \\ &\Delta(\lambda) = (\lambda - q)^{40}(\lambda + q^{-1})^{40}(\lambda - 1) \\ &m(\lambda) = (\lambda - q)(\lambda + q^{-1})(\lambda - 1). \end{aligned} \tag{2.12b}$$

Based on the results obtained for B_2, B_3 and B_4 we expect that for every standard solution of the B_j type there exist an additional $(2^{j-1} - 1)$ solutions of (2.3) where many of them are related by similarity transformations and simple changes of variable. We stress that the non-standard solutions discussed in this section do not follow the classical decomposition rule.

2.4. Link polynomials

The classification of knots and links constitutes an important problem of topology. As shown by Reidemeister, two knots (links) are combinatorially equivalent (can be deformed into each other) if and only if their diagrams can be transformed into one another by a set of three moves (Reidemeister moves) and planar isotropy [25]. What one wants is a method of distinguishing inequivalent knots and links. Link polynomials constitute such a distinguishing mechanism; they are mappings of combinatorially equivalence classes of knots (links) to the space of polynomials. Examples of such mappings are the well known Jones and Alexander–Conway link polynomials.

A theorem of Alexander [26] shows that there exist closed braids (opposite ends of a braid identified) in every combinatorially equivalence class of knots and (links), thus making braid theory relevant for the study of links and knots; the use of braid theory translates the topological problem of classifying equivalence classes of knots and links into an algebraic problem.

A theorem of Markov [27] states that two closed braids are combinatorially equivalent if they can be transformed into one another by successive application of the following moves:

$$\text{(I)} \qquad FG \to GF \qquad \forall F, G \in \mathbb{B}_n$$

$$\text{(II)} \qquad Fg_n^{\pm} \to F \qquad \forall F \in \mathbb{B}_n,\ Fg_n^{\pm} \in \mathbb{B}_{n+1}.$$

Link polynomials must therefore be mappings that are invariant under the above two moves.

Let $\mathbf{H}$ be a matrix such that

$$\mathbf{H} \equiv \mathbf{h}^{\otimes n} \in \mathrm{End}(V^{\otimes n})$$

where $\mathbf{h} \in \mathrm{End}(V)$ is an $N \times N$ diagonal matrix

$$(h)_i^j = h_i \delta_i^j. \tag{2.13}$$

The mapping [28]

$$L(F) = (\chi_+)^{-(n-1+\sigma)/2}(\chi_-)^{-(n-1-\sigma)/2}\,\mathrm{trace}(HF) \qquad F \in \mathbb{B}_n \tag{2.14}$$

is Markov invariant provided the following two conditions [5, 29] are satisfied:

$$\sum_{e=1}^{N} (S^{\pm})_{ie}^{je} h_e = \chi_{\pm}\delta_i^j \tag{2.15a}$$

$$S_{ij}^{ef}(h_i h_j - h_e h_f) = 0. \tag{2.15b}$$

Here σ is the sum of the exponents of the generators g_i in the braid word F and there is no summation over repeated indices in (2.15b). For every solution of (2.3) of the B_2, B_3 and B_4 types there exists a set $(\chi_+, \chi_-, \mathbf{h})$ for which conditions (2.15) are satisfied; results are given in table 1.

Table 1

	χ_+	χ_-	λ_1	λ_2	λ_3	$\mathbf{h}$
B_2						
S	1	q^{-8}	q	$-q^{-1}$	q^{-4}	$\mathrm{diag}(q^{-1}, q^{-3}, q^{-4}, q^{-5}, q^{-7})$
$\tilde{S}$	1	1	q	$-q^{-1}$	1	$\mathrm{diag}(q^{-1}, -q^{-1}, 1, -q, q)$
B_3						
S	1	q^{-12}	q	$-q^{-1}$	q^{-6}	$\mathrm{diag}(q^{-1}, q^{-3}, q^{-5}, q^{-6}, q^{-7}, q^{-9}, q^{-11})$
$\tilde{S}$	1	q^{-4}	q	$-q^{-1}$	q^{-2}	$\mathrm{diag}(q^{-1}, q^{-3}, -q^{-3}, q^{-2}, -q^{-1}, q^{-1}, q^{-3})$
B_4						
S	1	q^{-16}	q	$-q^{-1}$	q^{-8}	$\mathrm{diag}(q^{-1}, q^{-3}, q^{-5}, q^{-7}, q^{-8}, q^{-9}, q^{-11}, q^{-13}, q^{-15})$
$\tilde{S}$	1	q^{-8}	q	$-q^{-1}$	q^{-4}	$\mathrm{diag}(q^{-1}, q^{-3}, q^{-5}, -q^{-5}, q^{-4}, -q^{-3}, q^{-3}, q^{-5}, q^{-7})$
S^*	1	1	q	$-q^{-1}$	1	$\mathrm{diag}(q^{-1}, q^{-3}, -q^{-3}, -q^{-1}, 1, -q, -q^3, q^3, q)$

Define

$$L_\mu \equiv L(Fg_i^\mu) \qquad \mu \text{ integer } \forall F, g_i \in \mathbb{B}_n.$$

The link polynomials defined by (2.14) and table 1 obey the following skein relation:

$$L_{+3}-(\chi_-)^{1/2}(\lambda_1+\lambda_2+\lambda_3)L_{+2}+\chi_-(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)L_{+1}-(\chi_-)^{3/2}\lambda_1\lambda_2\lambda_3L_0=0. \quad (2.16)$$

3. Quantum *R* matrices associated with solutions of the B_2 and B_3 types

3.1. Baxterization

Given a solution S of (2.3) whose spectral decomposition is

$$S=\sum_{i=1}^{l}\lambda_i\mathbb{P}_i \quad (3.1)$$

where the λ_i are the l distinct eigenvalues of S, we seek a quantum R matrix $\check{R}(x)$ of the trigonometric type for which

$$\check{R}(x)=\sum_{i=1}^{l}\zeta_i(x)\mathbb{P}_i \quad (3.2)$$

$\zeta_i(x)=a_i+b_ix+c_ix^2+\ldots x^{l-1}$. The coefficients a_i, b_i, ... are constants which are determined by imposing the following constraints:

$$\begin{aligned}&\check{R}(x=0)=S\\&\check{R}(x=1)=\nu I \qquad \text{(initial condition)}\\&\check{R}(x)\check{R}(x^{-1})=\eta(x)I \qquad \text{(unitarity condition)}\end{aligned} \quad (3.3)$$

where ν is a constant and $\eta(x)$ some polynomial in x. For the case $l=3$, substituting (3.2) into (3.3) we get

$$\begin{aligned}&a_1=\lambda_1 \qquad a_2=\lambda_2 \qquad a_3=\lambda_3\\&a_1+b_1+c_1=a_2+b_2+c_2=a_3+b_3+c_3\\&a_1c_1=a_2c_2=a_3c_3 \qquad b_1(a_1+c_1)=b_2(a_2+c_2)=b_3(a_3+c_3)\\&a_1^2+b_1^2+c_1^2=a_2^2+b_2^2+c_2^2=a_3^2+b_3^2+c_3^2.\end{aligned} \quad (3.4)$$

There are many solutions to (3.4); the ones of interest here are those leading to the following formula [30]:

$$\begin{aligned}\check{R}(x)=&\left[\lambda_1+\left(\lambda_2+\frac{\lambda_1\lambda_3}{\lambda_2}\right)x+\lambda_3x^2\right]\mathbb{P}_1+\left(\lambda_2+(\lambda_1+\lambda_3)x+\frac{\lambda_1\lambda_3}{\lambda_2}x^2\right)\mathbb{P}_2\\&+\left[\lambda_3+\left(\frac{\lambda_1\lambda_3}{\lambda_2}+\lambda_2\right)x+\lambda_1x^2\right]\mathbb{P}_3\\=&\lambda_1\lambda_3x(x-1)S^{-1}+\lambda_3\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)xI-(x-1)S\end{aligned} \quad (3.5)$$

and to formulae obtained through all possible permutations of the three indices in (3.5); out of the six possibilities only three are distinct. Before proceeding any further a point should be made clear. The several cases examined in this paper clearly demonstrate that the conditions (3.3) are not sufficient to ensure that the construction (3.5) is a solution of (1.2*a*). All the new quantum R matrices reported below have been checked by direct substitution in (1.2*a*) using a symbolic manipulation computer code [24]. Our construction of formula (3.5) says nothing about the ordering of the eigenvalues. Our experience indicates that not all orderings lead to solutions of (1.2a); however, in some cases different orderings lead to different quantum R matrices for a given S matrix. We now illustrate these points by Baxterizing solutions of the B_2 and B_3 types.

3.2. Baxterizing solutions of the B_2 type

We first Baxterize the standard solution given in (2.6a). Using Jimbo's [14] formula (3.6), it may be verified that our formula (3.5) with the eigenvalue ordering $\lambda_1=q$, $\lambda_2=-q^{-1}$ and $\lambda_3=q^{-4}$ gives Jimbo's $B_2^{(1)}$ quantum R matrix while the ordering $\lambda_1=-q^{-1}$, $\lambda_2=q$ and $\lambda_3=q^{-4}$ leads to the $A_4^{(2)}$ result. The other distinct ordering, namely $\lambda_1=q$, $\lambda_2=q^{-4}$ and $\lambda_3=-q^{-1}$, does not give a solution of (1.2a). The non-standard solution $\tilde{S}$ given in (2.7a) may also be Baxterized in two distinct ways, each corresponding to a different ordering of the eigenvalues. Denoting the quantum R matrix corresponding to the ordering $\lambda_1=q$, $\lambda_2=-q^{-1}$ and $\lambda_3=1$ by $\check{R}_1(x;q)$, it follows from (3.5) that

$$\check{R}_1(x;q)=qx(x-1)(\tilde{S})^{-1}+(1-q)(q-q^{-1})xI-(x-1)\tilde{S}. \tag{3.6}$$

The explicit form of (3.6) was given in [22]. The quantum R matrix $\check{R}_2(x;q)$ corresponding to the ordering $\lambda_1=-q^{-1}$, $\lambda_2=q$ and $\lambda_3=1$ is

$$\check{R}_2(x;q)=qx(x-1)(\tilde{S})^{-1}+(1+q)(1-q^2)xI+q^2(x-1)\tilde{S}$$

where we have multiplied by an overall factor $-q^2$. Explicitly, $\check{R}_2(x;q)$ is as follows:

$$\check{R}_2(x,q)=\text{block diag}(T_1,T_2,T_3,T_4,T_5,T_{-4},T_{-3},T_{-2},T_{-1})$$

where the submatrices $T_{\pm i}$ are symmetric

$$T_1=T_{-1}=(x+q)(x-q^2) \qquad T_2=T_{-2}=\begin{pmatrix} x(1-q^2)(x+q) & q(x-1)(x+q)\\ & (x+q)(1-q^2)\end{pmatrix}$$

$$T_3=T_{-3}=\begin{pmatrix} x(1-q^2)(x+q) & 0 & q(x-1)(x+q)\\ & (x+q)(1-q^2x) & 0\\ & & (x+q)(1-q^2)\end{pmatrix}$$

$$T_4=T_{-4}=\begin{pmatrix} x(1-q^2)(x+q) & 0 & 0 & q(x-1)(x+q)\\ & x(1-q^2)(x+q) & q(x-1)(x+q) & 0\\ & & (x+q)(1-q^2) & 0\\ & & & (x+q)(1-q^2)\end{pmatrix} \tag{3.7}$$

$$T_5=\begin{pmatrix} \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5\\ & \omega_6 & \omega_7 & \omega_8 & \omega_9\\ & & \omega_{10} & \omega_{11} & \omega_{12}\\ & & & \omega_{13} & \omega_{14}\\ & & & & \omega_{15}\end{pmatrix}$$

with

$$\omega_1 = x[x(1-q)(q-q^{-1}) + q^{-1} - q^3] \qquad \omega_2 = -\mathrm{i}(q-q^{-1})x(x-1)$$
$$\omega_3 = -q^{1/2}(q-q^{-1})x(x-1) \qquad \omega_4 = \mathrm{i}q(q-q^{-1})x(x-1)$$
$$\omega_5 = q(x-1)(1+qx) \qquad \omega_6 = (1+q)(1-q^2)x[q^{-1}(x-1)+1]$$
$$\omega_7 = \mathrm{i}q^{1/2}(q-q^{-1})x(x-1) \qquad \omega_8 = (1-x)(x+q^3)$$
$$\omega_9 = \mathrm{i}q^2(q-q^{-1})(x-1) \qquad \omega_{10} = (qx+1)(x-q^2)$$
$$\omega_{11} = \mathrm{i}q^{5/2}(q-q^{-1})(x-1) \qquad \omega_{12} = q^{5/2}(q-q^{-1})(x-1)$$
$$\omega_{13} = (1+q)(1-q^2)[-q(x-1)+x] \qquad \omega_{14} = -\mathrm{i}q^3(q-q^{-1})(x-1)$$
$$\omega_{15} = (1-q^2)[q+q^2x-q^2+x].$$

The other distinct ordering of the eigenvalues ($\lambda_1 = q$, $\lambda_2 = q^{-6}$, $\lambda_3 = -q^{-1}$) does not give a solution of (1.2a).

3.3. *Baxterizing solutions of the B_3 type*

There are two distinct ways of Baxterizing the standard solution given in (2.8a). The formula (3.5) with the ordering $\lambda_1 = q$, $\lambda_2 = -q^{-1}$ and $\lambda_3 = q^{-6}$ gives the $\mathrm{B}_3^{(1)}$ quantum R matrix (our choice of signs in the centre block differs from that of Jimbo [14]) while the ordering $\lambda_1 = -q^{-1}$, $\lambda_2 = q$ and $\lambda_3 = q^{-6}$ gives Jimbo's $\mathrm{A}_6^{(2)}$ quantum R matrix. The non-standard solution $\tilde{S}$ given in (2.9a) may also be Baxterized in two distinct ways. The quantum R matrix corresponding to the ordering $\lambda_1 = q$, $\lambda_2 = -q^{-1}$ and $\lambda_3 = q^{-2}$ is

$$\check{R}_1(x;q) = qx(x-1)(\tilde{S})^{-1} + (1-q)(1-q^2)xI - q^2(x-1)\tilde{S}.$$
$$= \text{block diag}(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7, \Gamma_{-6}, \Gamma_{-5}, \Gamma_{-4}, \Gamma_{-3}, \Gamma_{-2}, \Gamma_{-1})$$

where the submatrices $\Gamma_{\pm i}$ are symmetric:

$$\Gamma_1 = \Gamma_{-1} = \omega_1 \qquad \Gamma_2 = \Gamma_{-2} = \begin{pmatrix} \omega_2 & \omega_3 \\ & \omega_4 \end{pmatrix} \qquad \Gamma_3 = \Gamma_{-3} = \begin{pmatrix} \omega_2 & 0 & \omega_3 \\ & \omega_1 & 0 \\ & & \omega_4 \end{pmatrix}$$

$$\Gamma_4 = \Gamma_{-4} = \begin{pmatrix} \omega_2 & 0 & 0 & \omega_3 \\ & \omega_2 & \omega_3 & 0 \\ & & \omega_4 & 0 \\ & & & \omega_4 \end{pmatrix} \qquad \Gamma_5 = \Gamma_{-5} = \begin{pmatrix} \omega_2 & 0 & 0 & 0 & \omega_3 \\ & \omega_2 & 0 & \omega_3 & 0 \\ & & \omega_5 & 0 & 0 \\ & & & \omega_4 & 0 \\ & & & & \omega_4 \end{pmatrix} \tag{3.8}$$

$$\Gamma_6 = \Gamma_{-6} = \begin{pmatrix} \omega_2 & 0 & 0 & 0 & 0 & \omega_3 \\ & \omega_2 & 0 & 0 & \omega_3 & 0 \\ & & \omega_2 & \omega_3 & 0 & 0 \\ & & & \omega_4 & 0 & 0 \\ & & & & \omega_4 & 0 \\ & & & & & \omega_4 \end{pmatrix} \qquad \Gamma_7 = \begin{pmatrix} \omega_6 & \omega_7 & \omega_8 & \omega_9 & \omega_{10} & \omega_{11} & \omega_{12} \\ & \omega_{13} & \omega_{14} & \omega_{15} & \omega_{16} & \omega_{12} & \omega_{17} \\ & & \omega_{18} & \omega_{19} & \omega_{20} & \omega_{21} & \omega_{22} \\ & & & \omega_{23} & \omega_{24} & \omega_{25} & \omega_{26} \\ & & & & \omega_{27} & \omega_{28} & \omega_{29} \\ & & & & & \omega_{30} & \omega_{31} \\ & & & & & & \omega_{32} \end{pmatrix}$$

with

$$\begin{aligned}
&\omega_1=(x-q^2)(x-q) && \omega_2=x(1-q^2)(x-q)\\
&\omega_3=q(x-1)(x-q) && \omega_4=(1-q^2)(x-q)\\
&\omega_5=(1-q^2x)(x-q) && \omega_6=(1-q)(1-q^2)x^2\\
&\omega_7=-q(q-q^{-1})x(x-1) && \omega_8=\mathrm{i}q(q-q^{-1})x(x-1)\\
&\omega_9=q^{3/2}(q-q^{-1})x(x-1) && \omega_{10}=-\mathrm{i}q^2(q-q^{-1})x(x-1)\\
&\omega_{11}=-q^2(q-q^{-1})x(x-1) && \omega_{12}=q(x-1)(qx-1)\\
&\omega_{13}=(1-q)(1-q^2)x[q^{-1}(1-x)+1] && \omega_{14}=-\mathrm{i}(q-q^{-1})x(x-1)\\
&\omega_{15}=-q^{1/2}(q-q^{-1})x(x-1) && \omega_{16}=\mathrm{i}q(q-q^{-1})x(x-1)\\
&\omega_{17}=-q(q-q^{-1})(x-1) && \omega_{18}=(1-q^2)x[x(1+q^{-1})-q-q^{-1}]\\
&\omega_{19}=\mathrm{i}q^{1/2}(q-q^{-1})x(x-1) && \omega_{20}=(1-x)(x-q^3)\\
&\omega_{21}=-\mathrm{i}q^2(q-q^{-1})(x-1) && \omega_{22}=\mathrm{i}q(q-q^{-1})(x-1)\\
&\omega_{23}=(qx+1)(x+q^2)-2qx(1+q) && \omega_{24}=-\mathrm{i}q^{5/2}(q-q^{-1})(x-1)\\
&\omega_{25}=-q^{5/2}(q-q^{-1})(x-1) && \omega_{26}=q^{3/2}(q-q^{-1})(x-1)\\
&\omega_{27}=(1-q^2)(q^2x-q-q^2+x) && \omega_{28}=\mathrm{i}q^3(q-q^{-1})(x-1)\\
&\omega_{29}=-\mathrm{i}q^2(q-q^{-1})(x-1) && \omega_{30}=(1-q)(1-q^2)[q(x-1)+x]\\
&\omega_{31}=-q^2(q-q^{-1})(x-1) && \omega_{32}=(1-q)(1-q^2).
\end{aligned}$$

Finally, the ordering $\lambda_1=-q^{-1}$, $\lambda_2=q$ and $\lambda_3=q^{-2}$ gives a different quantum R matrix

$$\check{R}_2(x,q)=-q^{-1}x(x-1)(\tilde{S})^{-1}+q(q-q^{-1})(q+q^{-2})xI-q^2(x-1)\tilde{S}.$$

We stress that all the new quantum R matrices given in section 3 have been verified by direct substitution in (1.2a) using a symbolic manipulation computer code. In all cases there were only two distinct ways of Baxterizing the braid group representations.

4. Concluding remarks

The results presented in this paper raise some interesting questions.

Remark 1. The underlying mathematical structure behind the standard solutions of (2.3) is the quantized universal enveloping algebra of simple Lie algebras. The fact that the non-standard solutions do not follow the classical decomposition rule hints at a different underlying mathematical structure. Recently the mathematical structure behind non-standard solutions of the A_n types [31, 32] has been identified (twisted quantum groups).

Remark 2. As yet we have no proof that the Baxterization formula (3.5) is valid in general; our experience in the many cases examined indicates that it probably is. Even if it is general our results show that not all orderings of the eigenvalues are permitted. Our construction of (3.5) clearly indicates that the constraints (3.3) are not sufficient to ensure that (3.5) is a solution. In that respect the work of Bazhanov [13] might shed some light on this problem. He has shown that a meromorphic function $R(\theta)$ $(x = e^{c\theta})$ yields a solution of (1.1) provided it satisfies, in addition to constraints equivalent to (3.3), the properties of automorphicity (quasi-periodicity) and crossing symmetry. It would be interesting to determine under what conditions our formula (3.5) satisfies that two extra constraints; this might shed some light on the ordering problem.

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